UNDERGRADUATE FOURTH SEMESTER (HONOURS) EXAMINATIONS, 2022

Subject: Mathematics **Course ID: 42112**

Course Code: SH/MTH/402/C-9 **Course Title: Multivariate Calculus**

Full Marks: 40 **Time: 2 Hours**

The figures in the margin indicate the full marks

Notations and symbols have their usual meaning

1. **Answer *any five* of the following questions:** $(\mathbf{2} \times \mathbf{5} = \mathbf{10})$

a) Show that $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+(x-y)^2}$ does not exist.

b) If $z = xe^{xy}$, where $x = t^2,\ y = \frac{1}{t}$, find $\frac{dz}{dt}$ at $t = 1$.

c) If $f(x,y) = x|x| + |y|$ for all $(x,y)\in\mathbb{R}^2$, then test the differentiability of f at the origin.

d) Find the directional derivative of $f(x,y,z) = xy + yz + zx$ along $(\hat{\imath} + 2\hat{\jmath} + \hat{k})$ at $(1,2,0)$.

e) Integrate $\iint r^2 \sin\theta\, dr d\theta$ over upper half of the circle $r = 2a\cos\theta$.

f) Show that $\frac{1}{2}\oint_c (Xdy - Ydx)$ represents the area bounded by the simple closed curve *C* on a plane.

g) Find the equation of the tangent plane to the surface $x^2 + y^2 + (z+1)^2 = 10$ at the point $(2,-1,2)$.

h) Show that $\vec{\nabla}\, r^n = n\, r^{n-2}\, \vec{r}$, where $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ and $r = |\vec{r}|$.

2. **Answer *any four* of the following questions:** $(\mathbf{5} \times \mathbf{4} = \mathbf{20})$

a) Find the extrema of the function f given by $f(x,y) = \sin x \sin y \sin(x+y)$ over E where $E = \{(x,y): x \geq 0, y \geq 0, x + y \leq \pi\}$.

b) If $U \subseteq \mathbb{R}^2$ and the function $f: U \to \mathbb{R}$ is such that only one of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exists and the other is continuous at a point $(a,b) \in U$ then prove that f is differentiable at (a,b).

c) Changing the order of integration prove that

$$\int_0^a \int_0^x \frac{\varphi'(y)dxdy}{\sqrt{(a-x)(x-y)}} = \pi[\varphi(a) - \varphi(0)].$$

d) Using polar transformation, evaluate

$$\int_0^\infty \int_0^\infty e^{-(x^2 + 2xy\cos\alpha + y^2)}\ dx\, dy$$

where $0 < \alpha < \frac{\pi}{2}$.

e) Show that

$$\iiint_V \frac{dx\, dy\, dz}{(1 + x + y + z)^3} = \frac{1}{2}\left[\log 2 - \frac{5}{8}\right]$$

where V is the region bounded by the coordinate planes $x = 0,\ y = 0,\ z = 0$ and the plane $x + y + z = 1$.

f) Use Stokes' theorem to show that

$$\oint_C (y\,dx + z\,dy + x\,dz) = -2\sqrt{2}\,\pi a^2$$

where C is the curve of intersection of the sphere $x^2 + y^2 + z^2 - 2ax - 2ay = 0$ and the plane $x + y = 2a$.

3. Answer *any one* of the following questions: $(\mathbf{10} \times \mathbf{1} = \mathbf{10})$

a) **(i)** Define repeated limits of a double variable function. Does the existence of the repeated limits imply the existence of the double limit? Justify your answer.

(ii) Compute the volume V, common to the ellipsoid of revolution $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and the cylinder $x^2 + y^2 - ay = 0$. 5+5

b) **(i)** Verify Green's theorem in the plane for

$$\oint_C [(3x^2 + 2y)dx - (x + 3\cos y)\,dy],$$

where C is the boundary of the region enclosed by the parallelogram having vertices at $(0,0), (2,0), (3,1)$ and $(1,1)$.

(ii) If $z = x\,f(x+y) + y\,g(x+y)$, prove that

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0.$$

where f and g are two differentiable functions.

iii) If $\vec{a}$ is a constant vector and $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$, prove that $(\vec{a}\cdot\vec{\nabla})\vec{r} = \vec{a}$.

5+3+2

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